Non-stationary Random Vibration Analysis of Vehicle with Fractional Damping

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Abstract
The viscoelastic damping is one of key factors that affect vehicle vibration. It has been demonstrated that fractional derivative model can accurately model this kind of frequency dependence damping very well. Generally, the excitation of rough road can be acted as a stationary random vibration process in time domain when a vehicle is traveling at constant speeds. But the road roughness should be considered as a non-stationary random vibration process when traveling at variable speeds. In this paper, the vehicle non-stationary process is investigated by using the fractional damping model. The road roughness is built using PSD (power spectral density) in space domain by Laplace transform. The proposed method is validated by simulating a vehicle running with variable speeds in time domain. The vehicle non-stationary random response in frequency domain are also acquired by FFT(Fast Fourier Transform) algorithm.

Keywords: non-stationary, random vibration, fractional damping, PSD

1 Introduction

The road profile, the only input of the passive suspension system, has a significant effect on the vehicle's vibration. Hence in order to generate a suitable road profile, it was investigated very carefully [1]. There are many possible ways to describe the road input. Generally, the road roughness is considered as a random process in space domain and if the velocity is constant, it can also be considered as a stationary random process in time domain.
The stationary random vibration method was adopted by many researchers to investigate the vehicle vibration response [2, 3]. The road roughness was acquired by a white noise signal with certain power spectral density (PSD). While previous study all assumed that the vehicles were running at certain constant speeds.

However, the vehicle always travels at variable speed, especially when starting, accelerating and breaking. Therefore, the vibration caused by rough road surface should be considered as a non-stationary random process [4, 5]. Fang [6] has constructed the models for two types of evolutionary random excitations: one may be obtained by filtering a stationary random process through a linear time-dependent system, while another may result from a nonlinear transformation of the argument of a stationary random process. Based on the models, the unified expressions of the response evolutionary power spectra are derived through the complex modal analysis. Hammond and White [7] started with a stationary process and then distorted the independent variable so that the process is "stretched out" or "compressed" so as to create a non-stationary process. Sobczyk [8] investigated stationary response to profile imposed excitation with randomly varying speed using perturbation methods. Sun [8] used state space approach and the transfer function approach to study the transient phenomenon subjected to a non-stationary vibration. Based on transient transfer function, Zhang [4] investigated a new time domain method by using the non-stationary excitation model of a rough road. In [5], instead of the conventional FFT (Fast Fourier Transform) method, the continuous wavelet transform as well as the discrete wavelet transform were applied to study the non-stationary inputs and responses of the vehicle vibration system.
Quarter-car model is very widely used to simulate the response of the vehicle subjected to non-stationary vibration by many researchers. The damping force was always modeled as a force which is proportional to the first derivative of the relative displacement between the sprung mass and the tire. However, in recent years, a number of papers have shown that viscoelastic damping can be modeled using fractional calculus, which can more accurately model the viscoelastic damping behavior, and can take frequency dependency into account. Caputo and Mainardi [10] introduced a memory-based damping model and showed that fractional derivatives and Mittag-Leffler function could be used to model viscoelastic phenomena. Rossikhin [11] showed that fractional derivatives could more accurately model the viscoelastic behavior of the damping. Bagley [12, 13] showed that a fractional damping relationship could be used to predicted transient structural response.
In this paper, based on transfer function method and the work done by Zhang [4], a new way for vehicle non-stationary random vibration research is proposed. A quarter-car model with fractional damping was built to analyze

the response of the vehicle subjected to the excitation of the road profile. Based on Laplace transform, the fractional order calculus was approximated by rational functions.

The paper is organized as follows: Section 2 introduces the fractional order calculus. In Section 3, the quarter-car model with fractional damping is introduced. Section 4 describes the construction of the non-stationary road roughness. The simulation of non-stationary vibration response is conducted and illustrated in Section 5. Section 6 offers our conclusions and further study.

2 Fractional Order Calculus

The idea of fractional order calculus has been known since the development of regular calculus, with the first reference probably being associated with Leibniz and L' Hospital in 1965. Even though the idea of fractional order operators is as old as the idea of the integer order. In the last decades, the use of fractional order operators and operations has become more and more popular among many research areas, such as in physical chemistry, electronics, mechanics, automatic control, robotics, signal processing, et al [14].

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). The most common definition of a fractional derivative is through Riemann-Liouville integral [15]:

$$ {}_cD_t^{\alpha} f(t) = \frac{1}{\Gamma(N-\alpha)} \frac{d^N}{dt^N} \int_c^t \frac{f(\tau)}{(t-\tau)^{\alpha-N+1}} d\tau \qquad (1) $$

Where $0 \le \alpha \le N \le \alpha + 1$, $N \in \mathbb{N}$. Here α is the arbitrary order of the derivative, $\mathbb{N} = \{1,2,3,\dots\}$ and the $\Gamma(\cdot)$ denotes the gamma function defined as $\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt$ for $n > 0$. Normally the lower integration limit c is chosen to be zero. The fractional order derivative definition is consistent with the definition of integer derivatives when $\alpha \in \mathbb{N}$. Modeling of viscoelastic behavior of damping normally results in the order of the time derivative being between zero and one and enabling the above equation to be rewritten in a simpler form:

$$ {}_cD_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_c^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau, 0 \le \alpha \le 1 \qquad (2) $$

For convenience, Laplace domain notion is usually used to describe the fractional integro-differential operation. The formula for the Laplace transform of the Riemann-Liouville fractional derivative has the form

$$ \int_0^{\infty} e^{-st} {}_0D_t^{\alpha} f(t) dt = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k {}_0D_t^{\alpha-k-1} f(0) \qquad (3) $$

for ($n-1 \le \alpha \le n$) where $F(s) = L[f(t)]$ is the normal Laplace transform.

If ${}_0D_t^{\alpha-k-1} f(0) = 0, k = 0,1,2,\dots n-1$ then

$$ L_0 D_t^{\alpha} f(t) = s^{\alpha} F(s) \qquad (4) $$

According to Oustaloup [14], it is possible to obtain useful approximations to calculate the fractional calculus by using of a frequency-band real non-integer differentiator:

$$ H(s) = s^{\alpha}, \alpha \in \mathbb{R}^+ \qquad (5) $$

Where α is a real non-integer. We give high and low transitional frequencies ω_h and ω_b, and the unit gain frequency and the central frequency of a band of frequencies ω_u ($\omega_u = \sqrt{\omega_h \omega_b}$) geometrically distributed around it.

The above function (5) can be approximated by a rational function:

$$ \hat{H}(s) = (\frac{\omega_u}{\omega_h})^n \prod_{k=-N}^{N} \frac{1 + s/\omega_k^{'}}{1 + s/\omega_k} \text{ with } n = \alpha \in \mathbb{R} \qquad (6) $$

using the following set of synthesis formulas:

$$ \omega_0^{'} = \alpha^{-0.5} \omega_u \qquad (7) $$

$$ \omega_0 = \alpha^{0.5} \omega_u \qquad (8) $$

$$ \frac{\omega_{k+1}^{'}}{\omega_k^{'}} = \frac{\omega_{k+1}}{\omega_k} = \alpha\eta > 1 \qquad (9) $$

$$ \frac{\omega_{k+1}^{'}}{\omega_k} = \eta > 0 \qquad (10) $$

$$ \frac{\omega_k}{\omega_k^{'}} = \alpha > 0 \qquad (11) $$

$$ N = \frac{\log(\omega_N / \omega_0)}{\log(\alpha\eta)} \qquad (12) $$

$$ \mu = \frac{\log \alpha}{\log(\alpha\eta)} \qquad (13) $$

More details about the approximated rational functions can be referred to [14].

3 Quarter-car Model with Fractional Damping

Quarter-car model has been widely used for suspension analysis and design by many researchers, because it is simple and can capture many important characteristics of the full model, and provide a suitable framework to investigate suspension control concepts [16].

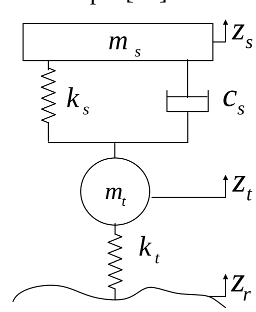

Figure 1: Quarter-Car Model

The quarter-car model, Figure 1, consists of a sprung mass supported on a suspension system which has stiffness and damping characteristics and the suspension system is connected to the unsprung mass. The tire is modeled as a spring. Traditionally, the stiffness characteristic was modeled as a force which is proportional to the relative displacement between the sprung mass and the tire, and the damping characteristic was modeled as a force which is proportional to the first derivative of the relative displacement between the sprung mass and the tire. However, in recent years, a number of papers have shown that viscoelastic damping can be modeled by using fractional calculus more accurately.

Generally, the dynamic equations of motion for the quarter model are given as follows:

$$m_s * \ddot{z}_s + k_s * (z_s - z_t) + c_s * (\dot{z}_s - \dot{z}_t) = 0 \quad (14)$$

$$m_t * \ddot{z}_t + k_t (z_t - z_r) - k_s * (z_s - z_t) - c_s * (\dot{z}_s - \dot{z}_t) = 0 \quad (15)$$

Where m_s is the sprung mass, which represents the car chassis, m_t is the sprung mass, which represents the wheel assembly, k_s and c_s are stiffness and damping of the uncontrolled suspension system, respectively; k_t is the vertical stiffness of the tire, z_s is the displacement of the sprung mass , z_t is the displacement of the tire, z_r is the road disturbance.

when fractional damping is introduced, the equations can be written as follows:

$$m_s * \ddot{z}_s + k_s * (z_s - z_t) + c_s * D_t^{\alpha} (z_s - z_t) = 0 \quad (16)$$

$$m_t * \ddot{z}_t + (z_t - z_r) - k_s * (z_s - z_t) - c_s * D_t^{\alpha} (z_s - z_t) = 0 \quad (17)$$

Where D_t^{α} represents the fractional differentiation operator, α is the calculus order, can be any real number. In practice, the value for α should be obtained by parameter identification method based on material experiment data [17]. While in this paper, the author just want to illustrate a new procedure for non-stationary research based on the fractional damping model and α is set to 2/3.

4 Simulation of Road Roughness

One of the most useful tools to describe the stationary road roughness is the power spectral density (PSD). When a car moves at a constant velocity u, the road roughness can be viewed as a stationary process in space domain, and the PSD of the road disturbance input can be expressed by

$$G_q(n) = G_q(n_0)(\frac{n}{n_0})^{-w} \quad (18)$$

Where $G_q(n)$ is the road PSD, n is the spatial frequency. The reference spatial frequency n_0 can be defined by $n_0 = 0.1(cycle/m)$, $G_q(n_0)$ is the road roughness coefficient, which is the value of PSD at the reference spatial frequency n_0, and represent different grades of road, w is called waviness and indicates whether the road has more long wavelengths (w is large) or short wavelengths (w is small). The waviness w is found within the range $1.75 \le w \le 2.25$, generally $w = 2$. When introduce spatial angular frequency Ω:

$$\Omega = 2\pi n \quad (19)$$

The equation (18) can be written as:

$$G_q(\Omega) = G_q(\Omega_0)(\frac{\Omega}{\Omega_0})^{-w} \quad (20)$$

Where $G_q(\Omega)$ is the road PSD, Ω is the spatial angular frequency (rad/m). Ω_0 is the reference spatial angular frequency, and $\Omega_0 = 2\pi n_0$. $G_q(\Omega_0)$ is also the road roughness coefficient, which is the value of PSD at the reference spatial angular frequency Ω_0. w has been mentioned before, and has the same definition.

When the vehicle drives at a constant velocity u, the relationship between the time frequency f and the vehicle forward velocity u is defined by:

$$f = u \times n \quad (21)$$

Derived from equation (21), the PSD of ground displacement has the following general form

$$G_q(f) = \frac{G_q(n_0) n_0^2 u}{f^2} \quad (22)$$

From the equation (22), we can get theoretic ground PSD under different frequencies.

The relationship between the spatial angular frequency Ω (rad/m) and the angular velocity ω (rad/s) can be defined as follows:

$$\omega = u\Omega \quad (23)$$

So the road roughness in frequency domain can be described as the follows:

$$G_q(\omega) = G_q(\Omega)/u = G_q(\Omega_0) u/\omega^2 \quad (24)$$

However, it may be troubles at $\omega = 0$, $G_q(\omega) = \infty$. An improved equation for PSD of road roughness in frequency domain is shown as,

$$G_q(\omega) = G_q(\Omega_0) u/(\omega^2 + \omega_0^2) \quad (25)$$

Where ω_0 is the lowest cut-off angular frequency, $\omega_0 = 2\pi f_0 = 2\pi u n_0$.

Equation (25) can be considered as a response of a first order linear system to white noise excitation.

Based on the theory of random vibration, the following relationship can be obtained,

$$G_q(\omega) = |H(\omega)|^2 S_w(\omega) \quad (26)$$

Where $H(\omega)$ is the transfer function, and S_w is the PSD of white noise, where normally $S_w(\omega) = 1$. So from the equation (25) and (26) the transfer function $H(\omega)$ can be described as:

$$H(\omega) = \frac{\sqrt{G_q(\Omega_0) u}}{\omega_0 + j\omega} \quad (27)$$

According to Laplace transform, the above equation can be written as:

$$H(s) = \frac{\sqrt{G_q(\Omega_0) u}}{\omega_0 + s} \quad (28)$$

The equation (28) can be viewed as the transfer function from white noise signal to road roughness. From the above equation, we can get the following equation:

$$\dot{z}_r(t) + \omega_0 z_r(t) = \sqrt{G_q(\Omega_0) u}\, w(t) \quad (29)$$

Where $z_r(t)$ is the road roughness, $w(t)$ is a white noise signal whose power spectral density is 1. Because $\omega_0 = 2\pi f_0 = 2\pi u n_0$, the above equation can be shown as:

$$\dot{z}_r(t) + 2\pi u n_0 z_r(t) = \sqrt{G_q(\Omega_0) u}\, w(t) \quad (30)$$

From equation (30), the road roughness can be obtained. The numerical calculation for the road roughness was carried out by Matlab/Simulink, as can be seen in Figure 2.

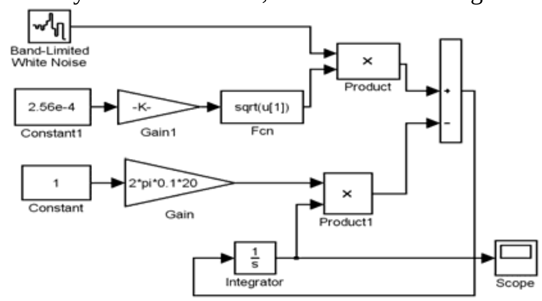

Figure 2: Simulink Block to Generate the Stationary Road Roughness

Figure 3 shows the time course of the stationary road roughness when vehicle drives at 20 *m/s* on Grade C road. FFT method is used to get the PSD from the time domain to the frequency domain. Figure 4 shows that the simulation road PSD fits the theoretical value very well, which illustrates the effectiveness of the method to get a stationary road roughness.

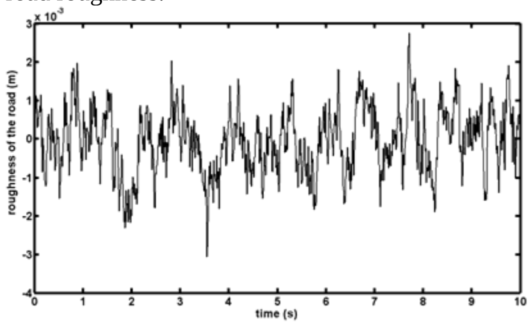

Figure 3: Time Course of Stationary Road

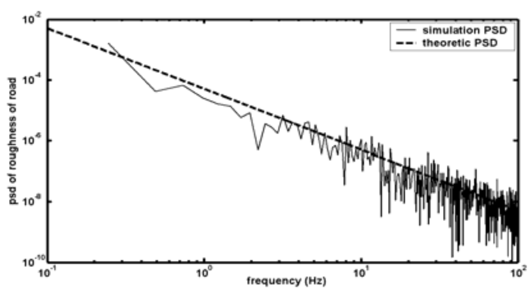

Figure 4: PSD of Stationary Road Roughness

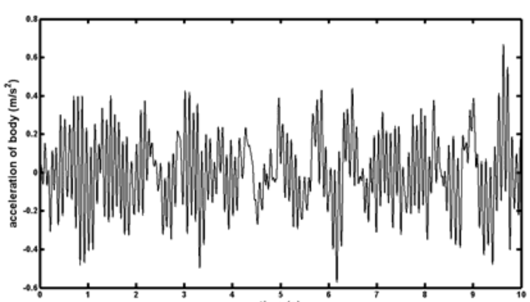

Figure5a: Stationary Response of the Body

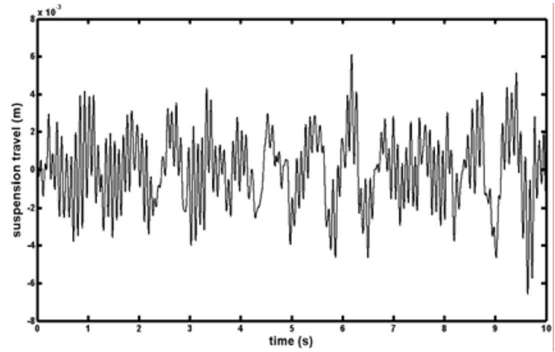

Figure5b: Stationary Response of the Suspension

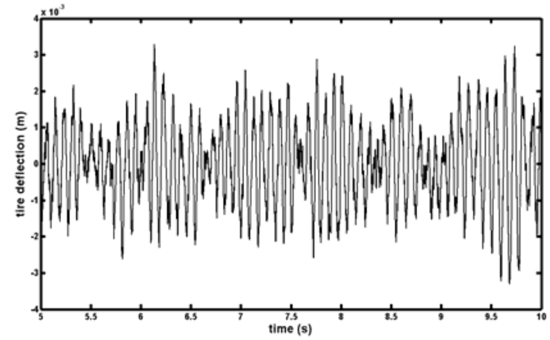

Figure5c: Stationary Response of the Tire

Figure5: Stationary response (20*m/s*)

As can be seen from the vehicle stationary response, figure 5, the body, the suspension and the tire responses to the stationary road are very regular and the maxim or minimum magnitudes of all the responses almost remain constant.

When a vehicle drives at variable speeds, the equation (30) can be expressed as:

$$\dot{z}_r(t)+2\pi u(t)n_0 z_r(t)=\sqrt{G_q(\Omega_0)u(t)}w(t) \qquad (31)$$

As same as the way to get a stationary random process, a non-stationary random process can be obtained from equation (31). The initial speed of the vehicle is 0 *m/s*, and then the vehicle will accelerate with the acceleration2 m/s^2 and 3 m/s^2, respectively. Figure 6 and Figure 8 show a velocity course of a non-stationary road. From Figures 6-9, we can see the roads and the PSD of the no-stationary road roughness under the used acceleration exhibit similar variation process.

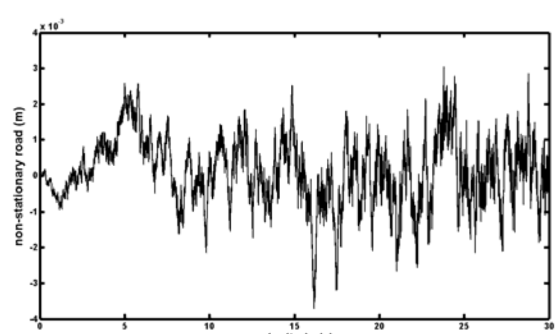

Figure 6: Velocity Course of Non-Stationary Road (2 m/s^2)

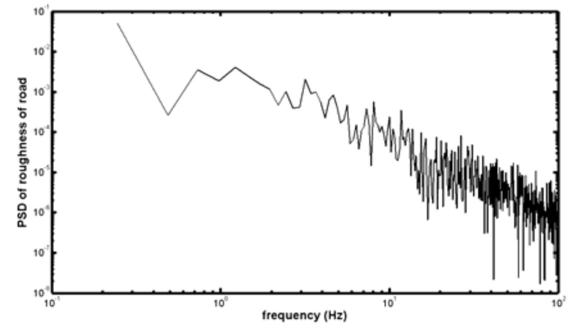

Figure 7: PSD of Non-Stationary Road Roughness (2 *m* /s^2)

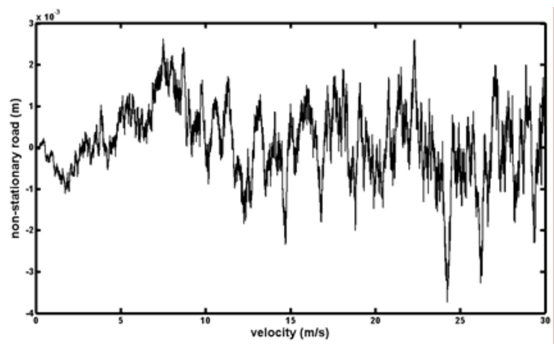

Figure 8: Velocity Course of Non-Stationary Road (3 *m* /*s*^{2})

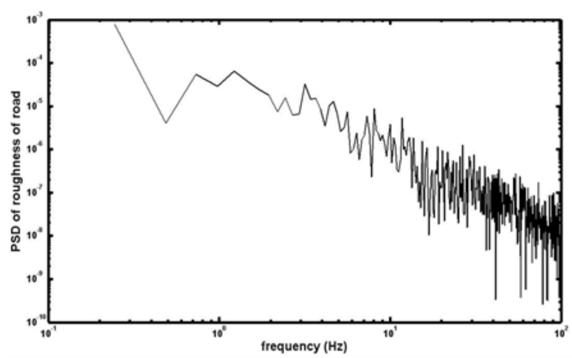

Figure 9: PSD of Non-Stationary Road Roughness (3 *m* /s^2)

5 Simulation of Non-stationary Response

The quarter-car model with fractional damping was built with Matlab/Simulink. The Figure 10 shows the flowchart of the system. The parameters of the quarter-car model used in the numerical calculations are given as follows: $m_s = 280kg$, $m_t = 36kg$, $k_s = 16000N/m$, $k_t = 160000N/m$, $c_s = 980N \cdot s/m$. The fractional order of the damper is set to 2/3.

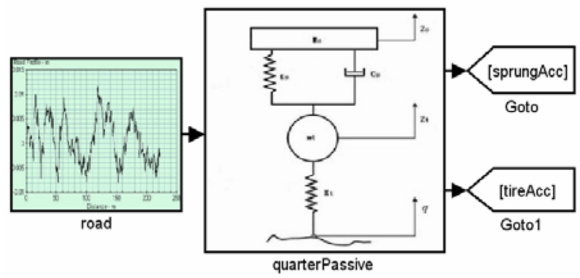

Figure 10: The Flowchart of the System in Matlab/Simulink

5.1 Time domain analysis

The initial speed of the vehicle is 0 *m/s*. The vehicle accelerates to 30 *m/s* with acceleration 2 *m/s*^{2} and 3 *m/s*^{2}, respectively. Figure 11 and Figure 12 show response to the non- stationary road roughness with vehicle acceleration 2 *m/s*^{2} and 3 *m/s*^{2}, respectively.

From Figure 11 and Figure 12, Compare to the stationary response, it can be seen that the amplitudes of the vehicle body's acceleration, the suspension travel displacement and the tire deflection all increase with speed. In fact, vehicles always drive at variable speeds, so it is not accurate to simulate the vehicle response subjected to stationary excitation, while assuming the vehicle drives at a constant speed. Comparing Figure 12 to Figure 11, we can seen the periodicity of all the response under acceleration 3*m/s*^{2} is smaller than the results under acceleration 2 *m/s*^{2}. Namely, it takes less time for a vehicle to reach same velocity with a big acceleration than that with a small acceleration. Furthermore, from the two simulation results, the RMS (root mean square) of the body acceleration is 0.1492 for acceleration 2 *m/s*^{2}, and the RMS of the body acceleration is 0.1650 for acceleration 3 *m/s*^{2}. So we can conclude that, the vibration is more severely when the vehicle drives with a larger acceleration.

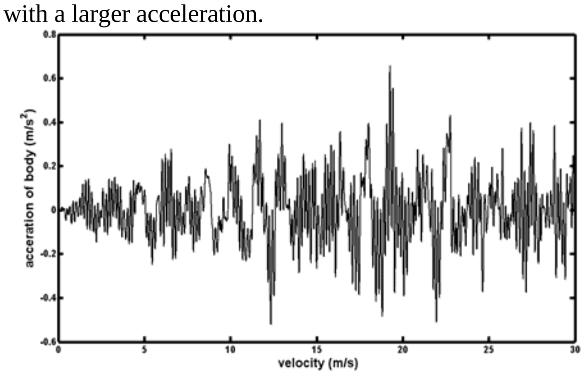

Figure11a: Non-Stationary Response of the Body

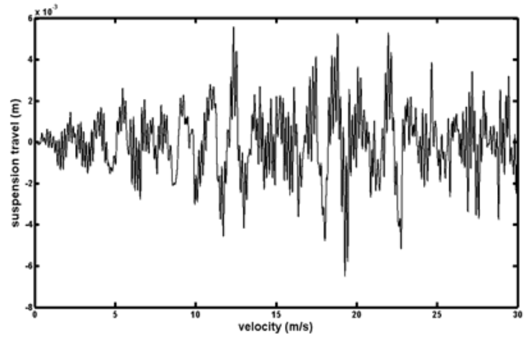

Figure11b: Non-Stationary Response of the Suspension

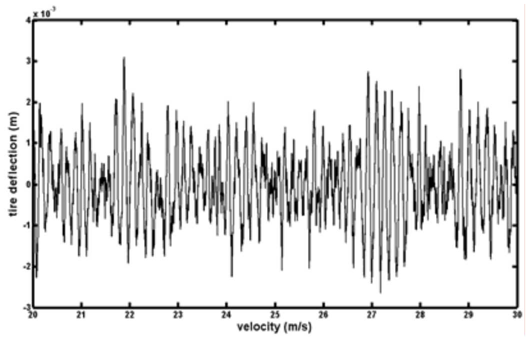

Figure11c: Non-Stationary Response of the Tire

Figure 11: Non-Stationary Response ($2m/s^2$)

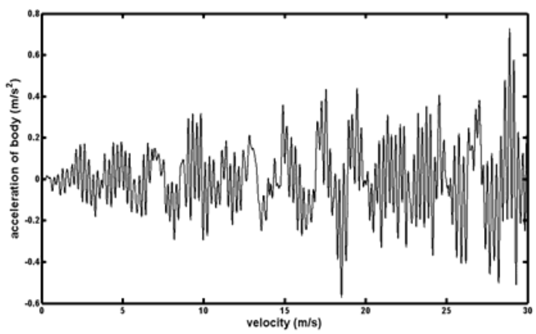

Figure12a: Non-Stationary Response of the Body

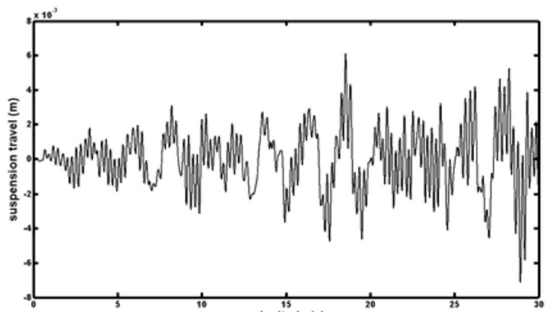

Figure12b: Non-Stationary Response of the Suspension

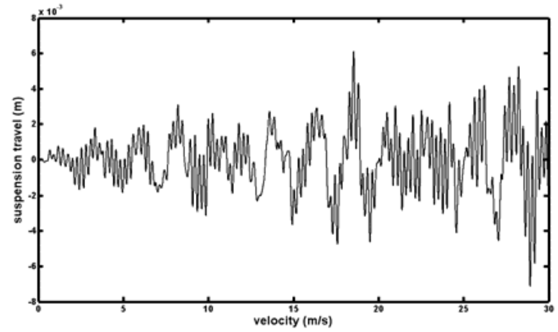

Figure12c: Non-Stationary Response of the Tire

Figure 12: Non-Stationary Response ($3m/s^2$)

5.2 Frequency domain analysis

In general, the ride comfort is frequency-sensitive. From the ISO2361 [3], the human body is very sensitive to vertical vibration in the frequency range of 4-8 Hz. Hence, it is necessary to evaluate the suspension subjected to non-stationary road excitation in frequency domain. The non-stationary random response in frequency domain of the vehicle are acquired by FFT(Fast Fourier Transform) Algorithm. Also, the vehicle initial speed is 0 *m/s* and it will accelerate with acceleration 2 m/s^2, 3 m/s^2 and 4 m/s^2,respectively. Figure 13 shows the frequency response of the body acceleration $\ddot{z}_s$, the suspension travel $z_s - z_u$, and the tyre displacement z_u. From Figure 13, we can see there are two response peaks for a quarter-car model and the maximum response values will increase with acceleration increasing. Of course, the quarter-car response is insensitive to the frequency range of 4-8 Hz.

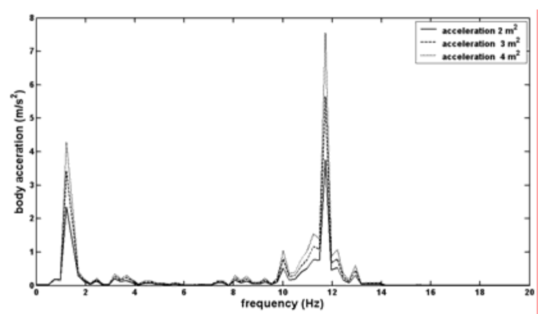

Figure13a: Frequency response of the Body

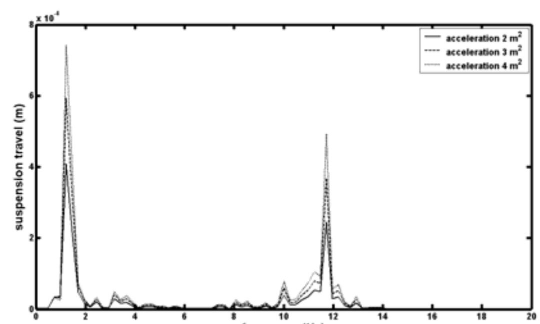

Figure13b: Frequency Response of the Suspension

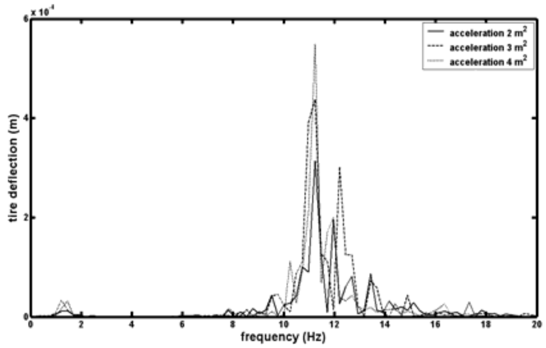

Figure13c: Frequency Response of the Tire

Figure 13: Frequency Response of Suspension

6 Conclusions

This paper proposed a new method for analyzing vehicle non-stationary random vibration. A quarter-car model with fractional damping was built to analyze the response of the vehicle subjected to the non-stationary excitation of the road profile. Based on Laplace transform, the fractional order calculus was approximated by rational functions. The comparison of the no-stationary vibration simulation results to the results of stationary vibration simulation results indicated the proposed method is effective.

However, the model built in the work is a linear system. The nonlinear system can be considered in the future. The model adopted in this paper is a quart-car model with two degrees of freedom. A much more complicated vehicle model with more freedom will be built to analyze the response subjected to non-stationary excitations. The parameter identification method can also be used to obtain the accurate fractional damping model in the future research.

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