Design, modeling and controllability of a spherical mobile robot

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Abstract

A spherical mobile robot, rolling on a plane with the help of two internal rotors and working on the principle of conservation of angular momentum has recently been fabricated in our group. The robot is a classic example of a nonholonomic system for which many existing algorithms do not easily apply. The objective is to study feasible path planning algorithms on this system. In this paper, we present the design details of the spherical robot fabricated along with the hardware used. We use Euler parameters which describe a unit quaternion for orientation of the robot and develop mathematical model to avoid singularity problem. We also prove controllability of the system in the quaternion space.

Keywords: Spherical robot, nonholonomic systems, Euler parameters.

1 Introduction

Mobile robotics is one of the important branches of robotics. For mobility of the robots, different motions like rolling, walking, hopping, sliding are used. Rolling motion has certain advantages over other motions. The problem of wearing and tearing is very less for the rolling motion. Also since the systems with rolling are nonholonomic, the set of configurations is reachable by lesser number of inputs. The role of friction is also conservative. For all these reasons, rolling motion is preferred in most of the cases in mobile robotics. In this paper, we present a detailed analysis of a spherical rolling robot. Due to the spherical shape, the robot recovers from the collisions with an unknown obstacles which is its intrinsic property. Due to this, it has practical applications like inspection and surveillance in hazardous environment. Any sensor can be mounted inside the spherical shell and the robot can be effectively used. It has an advantage of having one more orientation parameter as compared to car robots. As compared to single wheeled robots like gyroscopes [1], the spherical structure is statically stable. Also it is the classic example of a nonholonomic system for which existing path planning techniques do not apply. For all these features, we were interested in study of the robot and an autonomous spherical mobile robot has been designed and developed by our group.

A typical construction of the spherical mobile robot has a spherical shell with some internal driving unit(IDU) mounted inside the spherical shell. The robot in [2] is composed of a spherical shell and an arch shaped body. The driving unit consists of a pendulum and a controlling arch. The arched body and pendulum can control the pitch angle and the controlling arch below the pendulum controls the roll angle simultaneously with the pitch angle. In [3] and [4], the IDU consists of a wheel rolling inside the spherical shell which is in turn driven by a motor. The robot moves due to the disturbance of the system equilibrium due to unbalance of the inside construction. The 'Rollo', a spherical mobile robot designed and developed by P. Harmo et al.[5], the IDU is hanging from the rim. The rim can be rotated around the two axes of the IDU. The spherical shell rolls according to the movement of the rim. A small car like structure has been used as IDU in Sphericle [6]. The car has the unicycle kinematics and is driven with the help of two stepper motors. All the robots described up till now work on the principle of change of center of mass with the help of IDU for rolling the robot. Rollmob, designed and developed by L. Ferriere et al.[7] is a spherical ball driven by an universal wheel equipped with rollers. The rotation of the roller wheel drives the sphere around the direction parallel to the wheel axis, while the sphere rotates freely around the direction perpendicular to this axis. The construction of the robot designed by Bhattacharya et al.[8] is driven by a set of two mutually perpendicular rotors, attached to the spherical shell of the robot from inside. Along the Z axis there is a single rotor and along X axis there are two rotors which are rotated in tandem as a single rigid connected body. When rotors are rotated, the spherical robot rolls in the opposite direction due to the conservation of angular momentum. GroundBot, a spherical robot developed by Rotundus is designed for extraterrestrial exploration. The center of gravity of this robot is kept close to the ground with the help of a controlled pendulum. When it is raised, the ball rolls forward. When the pendulum is moved sideways, the ball turns. Spherobot, a spherical mobile robot by Ranjan Mukherji et al.[9] has radial spokes along which masses are placed. Radial movement of these masses create a moment about the center causing the motion of the robot. Cyclops [10] has two degrees of freedom in its locomotion system. It can pivot in place along its vertical axis and roll forward and backward along a fixed horizontal axis via a small motor and gear-head fixed inside. The review papers discussing constructional details of available Spherical Mobile Robots are [11, 12]. In this paper, we present a systematic study of a spherical mobile robot, designed, fabricated and analyzed in our laboratory. The organization of the paper is as follows. Section 2 describes the construction and design details of the robot. Section 3 describes the mathematical modeling of the system using quaternion. In Section 4 the controllability of the model is analyzed in the quaternion space. Section 5 provides concluding remarks.

2 Design

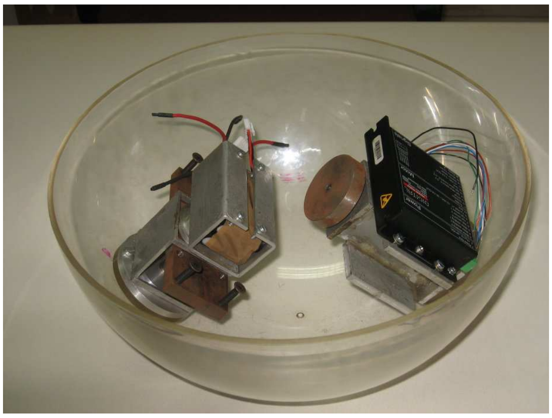

Figure 1: Placing of components in a half shell

The spherical mobile robot designed in our laboratory works on the principle of conservation of angular momentum. The robot has two internal rotors similar to a few constructions mentioned above. The robot's spherical shell is made up of acrylic material having 4 mm thickness. The inner radius of the robot is 30 cm. A crucial aspect of the design is to place the internal components such that the center of mass of the robot is exactly at the geometric center of the sphere. This is very important so that the robot will not tip over on its own. The easiest way to achieve this is to place all the parts symmetrically. As will be seen later, the robot is controllable with only two inputs namely the speeds of the rotors placed along two mutually perpendicular rotors. We, therefore, consider two rotors along X and Z axis of the body frame. These rotors are driven by two D.C motors MAXON EC $\phi 32$, Brushless, $80W$. Two batteries of the type Polyquest PQ08003 TWENTY LIPO PACK of capacity 800mAh each, are used for supplying power to one motor. So in all there are 4 batteries. There are two speed control units to control the speed of the motors which receive control signals from the external controller such as personal computer through wireless communication. For symmetrically placing the

Figure 2: Construction of the spherical robot

components, the motor along with rotor and one speed control unit is placed on one side and the battery along with dead weight is placed on the diametrically opposite direction as shown in Figure 2. Similarly another pair of motor assembly and battery+dead weight are placed in diametrically opposite directions. The robot is fabricated as two hemispheres as shown in Figure 1. Each hemisphere consists of one motor assembly and one battery+dead weight assembly. It is absolutely critical that there be no relative motion between the two hemispheres while in motion. The total weight of the robot is 3.4 KG.

3 Mathematical Modeling

This section describes the development of an analytical model of the spherical rolling robot using quaternion. Consider a spherical robot rolling on a horizontal plane as shown in the figure (3). An inertial coordinate frame is attached to the surface and denoted as XYZ with its origin at a point O. The body coordinate axes xyz are attached to the sphere and have their origin at the center of the sphere G. The set of generalized coordinates

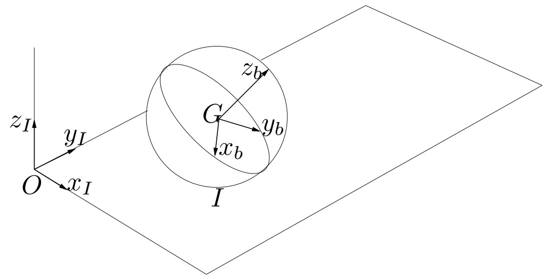

Figure 3: Sphere rolling on a surface

describing the sphere consists of [13]

- Coordinates of the contact point I on the plane.
- Any set of variables describing the orientation of the sphere.

We use Euler parameters (instead of Euler angles) which is a set of 4 parameters to describe the orientation of the sphere. Euler parameters have the advantage of being a nonsingular two to one mapping with the rotation. In addition, Euler parameters form a unit quaternion and can be manipulated using quaternion algebra[14],[15], [16], [17]. Using a set of Euler parameters for defining orientation, we get the set of generalized coordinates as

$$p = (x, y, e_0, e_1, e_2, e_3)^T,$$

where e_0, e_1, e_2 and e_3 are the Euler's parameters describing orientation forming unit quaternion such that

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1.$$

(x, y) are the coordinates of the contact point I. Let i_b, j_b, k_b be the unit vectors of the body frame and i_I, j_I, k_I be the unit vectors of the inertial frame. The inertial axes can be transformed to body axes using[15]

$$\begin{bmatrix} i_b \\ j_b \\ k_b \end{bmatrix} = 2 \times T \begin{bmatrix} i_I \\ j_I \\ k_I \end{bmatrix}, \tag{1}$$

where

$$T = \begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 + e_0e_3 & e_1e_3 - e_0e_2 \\ e_1e_2 - e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 + e_0e_1 \\ e_1e_3 + e_0e_2 & e_2e_3 - e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}.$$

If ω is the angular velocity of the sphere given by

$$\omega = \omega_x i_b + \omega_y j_b + \omega_z k_b. \tag{2}$$

The projection of the angular velocity vector on the body axes can be related to the rate of change of the Euler parameters w.r.t. time using the relationship given in [15][18]

$$\begin{bmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 2 \times \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \\ -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & -e_1 & e_0 \end{bmatrix} \begin{bmatrix} \dot{e_0} \\ \dot{e_1} \\ \dot{e_2} \\ \dot{e_3} \end{bmatrix}. \tag{3}$$

3.1 No Slip Constraint

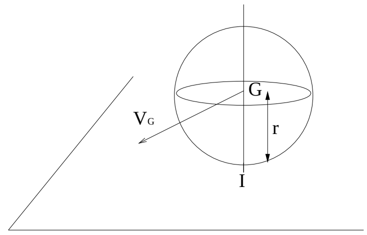

Figure 4: Sphere rolling with no slip

During pure rolling process, the sphere moves without slipping. As it rolls, it is assumed to rotate instantaneously about the axis passing through the point of contact I. So the instantaneous velocity of the contact point I with respect to the inertial coordinates is zero

$$V_I = 0. \tag{4}$$

We consider the velocity of the center of the sphere as

$$V_G = \dot{x} i_I + \dot{y} j_I, \tag{5}$$
$$V_G = V_I + \omega \times \bar{r}, \tag{6}$$

where r is the radius vector of the sphere from I to G given by

$$\bar{r} = r k_I.$$

From (4),(5),(6) we get

$$\dot{x} i_I + \dot{y} j_I = \omega \times r k_I.$$

Using the expression for ω from (2) and (3) and expressing body vectors in terms of inertial vectors using (1) we get

$$\begin{aligned} \dot{x} + 2re_2\dot{e}_0 - 2re_3\dot{e}_1 - 2re_0\dot{e}_2 + 2re_1\dot{e}_3 = 0. \\ \dot{y} - 2re_1\dot{e}_0 + 2re_0\dot{e}_1 - 2re_3\dot{e}_2 + 2re_2\dot{e}_3 = 0. \end{aligned} \tag{7}$$

For a unit sphere, the no slip constraint equations reduce to

$$\begin{aligned} \dot{x} + 2e_2\dot{e}_0 - 2e_3\dot{e}_1 - 2e_0\dot{e}_2 + 2e_1\dot{e}_3 = 0. \\ \dot{y} - 2e_1\dot{e}_0 + 2e_0\dot{e}_1 - 2e_3\dot{e}_2 + 2e_2\dot{e}_3 = 0. \end{aligned} \tag{8}$$

3.2 First order model of the sphere rolling on a plane

We assume the sphere to have unit radius without any loss of generality. If angular velocities of the sphere about body co-ordinate axes are ω_x, ω_y, ω_z, then the equations (3) and (8) describe the kinematics of the sphere fully giving a set of state equations as

$$Q \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{e_0} \\ \dot{e_1} \\ \dot{e_2} \\ \dot{e_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \omega_x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \omega_y + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega_z, \tag{9}$$

where

$$Q = \begin{bmatrix} 1 & 0 & 2e_2 & -2e_3 & -2e_0 & 2e_1 \\ 0 & 1 & -2e_1 & 2e_0 & -2e_3 & 2e_2 \\ 0 & 0 & 2e_0 & 2e_1 & 2e_2 & 2e_3 \\ 0 & 0 & -2e_1 & 2e_0 & 2e_3 & -2e_2 \\ 0 & 0 & -2e_2 & -2e_3 & 2e_0 & 2e_1 \\ 0 & 0 & -2e_3 & 2e_2 & -2e_1 & 2e_0 \end{bmatrix}.$$

It can be shown that the matrix Q is an orthogonal matrix and hence is invertible. The internal rotors (which are actuators for the robot) are located along X and Z axis of the body frame. The robot is symmetric in construction and we therefore consider $\omega_y = 0$, reducing the system equations from (9) to

$$\dot{p} = X_1(p)\omega_x + X_2(p)\omega_z, \tag{10}$$

where

$$p = \begin{bmatrix} x & y & e_0 & e_1 & e_2 & e_3 \end{bmatrix}^T. \tag{11}$$

$$X_1(p) = Q^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(e_0e_3 + e_2e_1) \\ (e_3^2 + e_2^2 - e_0^2 - e_1^2) \\ -\frac{1}{2}e_1 \\ \frac{1}{2}e_0 \\ \frac{1}{2}e_3 \\ -\frac{1}{2}e_2 \end{bmatrix}.$$

$$X_2(p) = Q^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2(e_2e_3 - e_0e_1) \\ -2(e_1e_3 + e_0e_2) \\ -\frac{1}{2}e_3 \\ \frac{1}{2}e_2 \\ -\frac{1}{2}e_1 \\ \frac{1}{2}e_0 \end{bmatrix}.$$

3.3 Conservation of angular momentum

As shown in Figure 2, the construction of the robot is symmetrical. The gravitational force therefore passes through the center of the sphere G. Reaction force also passes though G as the point is exactly above the point of contact I. Since there exist no dissipative frictional forces, the sum of external moments at the point G are zero. The angular momentum at the center of the sphere therefore is a conserved quantity, giving

$$\dot{H}_G = 0.$$

We further assume that the sphere starts at rest giving

$$H_G(t_0) = 0 \;\Rightarrow\; H_G(t) = 0,\; \forall\, t \geq t_0.$$

We introduce two new variables ψ_1 and ψ_2 for the rotor angles of X and Z rotor respectively. The sphere is assumed to be symmetrical in construction about the body axes resulting in diagonal inertia matrices for the sphere shell as well as for X rotor and Z rotor. Let I^s, J^x and J^z be the inertia matrices of the sphere, the X rotor and the Z rotor respectively with respect to the body axes given by

$$I^s = \begin{bmatrix} I^s_{xx} & 0 & 0 \\ 0 & I^s_{yy} & 0 \\ 0 & 0 & I^s_{zz} \end{bmatrix},$$

$$J^x = \begin{bmatrix} J^x_{xx} & 0 & 0 \\ 0 & J^x_{yy} & 0 \\ 0 & 0 & J^x_{zz} \end{bmatrix},$$

$$J^z = \begin{bmatrix} J^z_{xx} & 0 & 0 \\ 0 & J^z_{yy} & 0 \\ 0 & 0 & J^z_{zz} \end{bmatrix}.$$

The angular momentum at G can be written as

$$H_G = (I_x\omega_x)i_b + (I_y\omega_y)j_b + (I_z\omega_z)k_b + J^x_{xx}(\omega_x + \dot{\psi}_1)i_b + J^z_{zz}(\omega_z + \dot{\psi}_2)k_b = 0, \tag{12}$$

where

$$I_x \triangleq I^s_{xx} + J^z_{xx}.$$
$$I_y \triangleq I^s_{yy} + J^x_{yy} + J^z_{yy}.$$
$$I_z \triangleq I^s_{zz} + J^x_{zz}.$$

The inputs for the kinematic model which are components of the angular velocity vector in the body frame are related to the rotor speeds $\dot{\psi}_1$ and $\dot{\psi}_2$ by the relation of the conservation of angular momentum (12).

3.4 Properties of the model

3.4.1 Controllability

Before we proceed to path planning of the spherical robot, it is essential to check whether there exists a path that connects any arbitrary configurations of the sphere. The question can be answered using a result known as the *Chow's Theorem*[19]. In this section, we use the algorithm given in [20] to answer the question. Consider the system described by the equation (10).

$$\dot{p} = X_1(p)\omega_x + X_2(p)\omega_y,$$

where p, $X_1(p)$ and $X_2(p)$ are given by (11). We compute the following Lie Brackets using a Philip Hall convention [19], [20]

$$X_3(p) = [X_1, X_2].$$
$$X_4(p) = [X_1, X_3].$$
$$X_5(p) = [X_2, X_3].$$
$$X_6(p) = [X_1, X_4].$$

A distribution is formed using the above generated Lie Bracket vector fields given by

$$\Delta = [X_1, X_2, X_3, X_4, X_5, X_6],$$

where

$$X_3 = \begin{bmatrix} 2e_1^2 - 2e_0^2 + 2e_3^2 - 2e_2^2 \\ 4(e_2e_1 - e_0e_3) \\ \frac{1}{2}e_2 \\ \frac{1}{2}e_3 \\ -\frac{1}{2}e_0 \\ -\frac{1}{2}e_1 \end{bmatrix}.$$

$$X_4 = \begin{bmatrix} 6(e_0e_1 - e_2e_3) \\ 6(e_1e_3 + e_0e_2) \\ \frac{1}{2}e_3 \\ -\frac{1}{2}e_2 \\ \frac{1}{2}e_1 \\ -\frac{1}{2}e_0 \end{bmatrix}.$$

$$X_5 = \begin{bmatrix} 6(e_0e_3 + e_2e_1) \\ 3(e_3^2 + e_2^2 - e_0^2 - e_1^2) \\ -\frac{1}{2}e_1 \\ \frac{1}{2}e_0 \\ \frac{1}{2}e_3 \\ -\frac{1}{2}e_2 \end{bmatrix}. \qquad (13)$$

$$X_6 = \begin{bmatrix} 4(-e_1^2 + e_0^2 - e_3^2 + e_2^2) \\ 8(-e_1e_2 + e_0e_3) \\ -\frac{1}{2}e_2 \\ -\frac{1}{2}e_3 \\ \frac{1}{2}e_0 \\ \frac{1}{2}e_1 \end{bmatrix}.$$

It can be observed that all higher order brackets can be expressed in terms of X_1, X_2, X_3, X_4, X_5 giving the rank of the distribution as 5. As the Euler parameters are used, the number of the state variables is 6. We are using a system of 4 parameters to describe the orientation. But all possible orientations lie on a hyper surface defined by a level set

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1, \qquad (14)$$

which is a 3 sphere. This gives the dimension of the configuration space as 5 which is equal to the rank of the distribution. So using *Chow's Theorem* the system is controllable and can be taken from any position to any arbitrary position using Lie Bracket motions described by the vector fields (13).

3.4.2 Nilpotency

It can be observed that higher order Lie products are not zero so the system is not nilpotent.

3.5 Conversion into a chained form

For determining the degree of nonholonomy, we construct a distribution associated with the control system (10) as

$$\Delta = \text{span}\{X_1, X_2\}.$$

We then construct filtrations associated with the distribution Δ as

$$\begin{aligned} E_0 &= \Delta, \quad F_0 = \Delta, \\ E_1 &= E_0 + [E_0, E_0], \quad F_1 = F_0 + [F_0, F_0], \\ E_2 &= E_1 + [E_1, E_1], \quad F_2 = F_1 + [F_1, F_0], \\ E_3 &= E_2 + [E_2, E_2], \quad F_3 = F_2 + [F_2, F_0], \\ E_4 &= E_3 + [E_3, E_3], \quad F_2 = F_3 + [F_3, F_0]. \end{aligned}$$

According to [21], a feedback transformation which puts a driftless two input nonholonomic system 9 into a chained form exists if and only if

$$\dim(E_i) = \dim(F_i) = i + 2; \qquad i = 0, \ldots, n-2. \quad (15)$$

In this particular case

$$\begin{aligned} &\text{For} \quad i = 0, \quad \dim(F_0) = 2, \\ &\text{For} \quad i = 1, \quad \dim(F_1) = 3, \\ &\text{For} \quad i = 2, \quad \dim(F_2) = 5 \neq i + 2, \\ &\text{For} \quad i = 3, \quad \dim(F_3) = 5, \end{aligned}$$

It can be observed that the condition (15) is not satisfied and it is not possible to convert the model into a chained form.

3.6 The degree of nonholonomy and growth vector

It can be observed from the way the filterations grow that the degree of nonholonomy of the system is 3 with the growth vector $(2, 3, 5)$ and the relative growth vector $(2, 1, 2)$.

4 Conclusion

Design and constructional features of a spherical mobile robot rolling on a plane are presented in this paper. Kinematic model of the system is developed using quaternion for description of the orientation of the robot. It can be observed that the model is nonsingular and valid everywhere. It is shown that the system is fully controllable and can be taken from any arbitrary configuration to any arbitrary configuration within unit 3-sphere in the quaternion space. The system is nonholonomic with the degree of nonholonomy 3 and growth vector $(2, 3, 5)$. It is not nilpotent and also can not be converted into 'a chained form'. It, therefore represents a class of systems for which all existing path planning techniques fail to apply. Dedicated path planning techniques can be developed using quaternion model developed which is future scope of our research.

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