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Dynamic modeling and tracking control of a four wheeled nonholonomic mobile robot

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Abstract

This paper presents methodologies for kinematic/dynamic modeling and trajectory tracking for a four wheeled nonholonomic mobile robot. The vehicle with two front (steering) and two rear (driving) wheels is considered. The complete dynamic model of such a wheeled mobile robot is established using the Euler's Lagrange equation and MATHEMATICA. Then a dynamical extension that makes possible the integration of a kinematic controller and a torque controller is presented .A combined kinematic/torque control law is developed using backstepping approach and asymptotic stability is guaranteed by Lyapunov theory. The mobile robot is modeled as a non holonomic system subject to pure rolling, no side slipping constraints. Simulation results are performed to illustrate the efficacy of the proposed control strategy.

Key words: mobile robot, backstepping control, nonholonomic systems, trajectory tracking, Lyapunov stability.

1 Introduction

There is a wide literature concerning control of nonholonomic mobile robots published in last fifteen years. A mobile robot is one of the well known system with nonholonomic constraints and there are many works on its tracking control .

The majority of research effort in the literature has concentrated on the use of kinematic model of the vehicle (where the velocities are the inputs),and less research has been done to solve the problem of integrating the nonholonomic kinematic controller with the dynamics of the systems. These mobile robots have applications in industrial, household, military, security, space and office automation. For nonholonomic systems such as mobile robots their kinematic constraints make time derivative of some configuration variables nonintegrable (Xiaoping, Y. & Yamamoto,Y., 1996). Due to the appearance of the nonholonomic constraints the motion planning and the tracking control of mobile robots are difficult to be managed. In the phase of motion planning (Wilson, D. E., and Luciano, E. C., 2002) a suitable trajectory is designed to connect the initial posture (i.e. the position and orientation of the robot) and the final one such that no collisions with obstacles would occur and kinematics constraints are satisfied. In this paper we study the kinematics and dynamics of the nonholonomic systems such that every path can be followed efficiently.Several control solutions for trajectory tracking for mobile robots have been proposed,as for example, Lyapunov direct method (Kanayama et al., 1990; Samson, 1993).The idea of input-output linearization was further explored by (Oelen, W. & Amerongen ,J., 1994) for a two degree of freedom mobile robot. The class of nonholonomic system in chained form was introduced by (Murray, R. M., & Sastry, S. S.,1993) and has been studied as a bench mark example by several authors. It is well known that many mechanical system with nonholonomic constraints can be locally or globally, converted to chained form under coordinate change and state feedback. Interesting examples of such mechanical systems include tricycle-type mobile robots, cars towing several trailers, the knife edge(Murray, R. M., & Sastry, S. S., 1993), (Kolmanovsky, I. & McClamroch, N. H. ,1995). Trajectory planning algorithm for a four-wheel-steering (4WS) vehicle based on vehicle kinematics was introduced by (Danwei, W. & Feng, Q., 2001). A new analytical solution to mobile robot trajectory generation in the presence of moving obstacles for a four wheel mobile robot based on its kinematics was introduced by Zhibua, Q., Wang, J. &Clinton, E. P., 2004).Trajectory tracking control of tri-wheeled mobile robots in skew chained form system was introduced by(Tsai, P. S., Wang, L. S., & Chang, F. R., 2006).

In this paper a feedback velocity following control law is designed such that the mobile robot velocities converge asymptotically to the given velocity inputs. Finally this second control signal is used by the computed – torque feedback controller to compute the required torques for the actual mobile robots. Although several approaches using the lagrange's equation of motion with multipliers have been proposed in modeling of mobile robots, their derivation procedures of finding dynamic models are too complicated and time consuming. To circumvent the difficulty ,this paper contributes a direct system modeling approach for the four wheel nonholonomic mobile robot through the use of the commercial package MATHEMATICA. By virtue of the advantages of the symbolic computation in MATHEMATICA the dynamic model of the four wheeled nonholonomic mobile robot can be easily developed and then represented in a state space form. With the proposed nonlinear tracking control law, the entire state of the system to asymptotically track to the desired trajectory is definitely ensured.

2 System Modelling

This section presents the complete governing equation of the four wheel mobile robot system, investigates the structural properties of the derived models and validates the developed model in comparison with other's well known work.

2.1 System Description

Consider the four wheeled mobile robot as shown in figure 1. The mobile robot under consideration has two rear driving wheels driven independently by two DC servomotors and the front two steering wheels.
To simplify the modeling derivation of the wheeled mobile robot under consideration the following assumption are made.
(1) The wheeled mobile robot is built from rigid mechanism .
(2) There is zero or one steering link per wheel.
(3) All steering axes are perpendicular to the surface of motion .
(4) The surface is a smooth plane.
(5) No slip occurs between the wheel and the floor.
The following notation will be used in the formulation of the constraint equations and the motion equations of the wheeled mobile robot.

P_o	the centre of the mobile plateform.
c	distance between the front wheel axle and the plateform centre of gravity P_o.
d	the distance between the rear wheel axle and platform centre of gravity P_o.
2b	the wheel span.
r	the radius of each wheel.
m_c	the mass of the plateform without the driving Wheels and the rotors of the DC motors.
m_w	the mass of each driving wheel with its motor.
I_c	the moment of inertia of the plateform without the driving wheels and the motors about a vertical axis through P_o.
I_m	the moment of inertia of each wheel about the wheel axis.
I_w	the moment of inertia of each wheel about the wheel diameter.

3 Kinematics of the mobile robot

The four wheel mobile robot considered in this paper is shown in fig.1, its front wheels are steering wheels, and its rear wheels are driving wheels. The distance between the front wheel axle and plateform centre of gravity is c and distance between the rear wheel axle and plateform centre of gravity is d and 2b is the wheel span.The trajectory planning will be done for the plateform centre of gravity. Let the generalized co-ordinates be $q=[x_0, y_0, \phi_0, \phi]^T$, where (x, y) are the cartesian coordinates of the centre of gravity of the mobile plateform with respect to co-ordinate frame $\{U\}$. The four wheels are located at p_1, p_2, p_3 and p_4 on the mobile platform respectively and p_c is the centre of the mobile platform. Six co-ordinate frames are defined for describing position and orientation of the mobile robot - $\{1\}$ is the frame fixed on wheel 1 with x_1 - axis is chosen to be along the horizontal radial direction and y_1 axis in the lateral direction .Likewise $\{2\},\{3\}$ and $\{4\}$ are the frame defined for the wheel 2, 3 and 4 respectively and $\{0\}$ is the frame defined at point p_c. The orientation of the vehicle body is characterized by ϕ_0 which is the angle from x_U to x_0 . ϕ_1 and ϕ_2 are the front two steering angle and ϕ is considered as a virtual steering angle of a virtual wheel which is at the middle point of the front two steering wheels i.e. the angle at which the whole plateform changes the orientation due to steering angles ϕ_1 and ϕ_2 of the front two steering wheels.

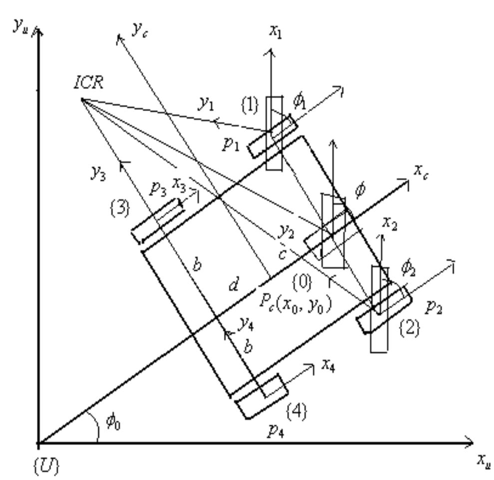

Figure 1: Four wheel mobile robot and co-ordinate frame

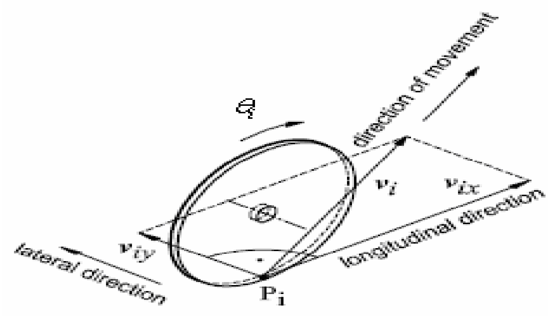

Figure 2: Velocities of wheels

To develop the kinematic model of the wheeled mobile robot ,the *ith* wheel is considered as rotating with angular velocity $\dot{\theta}_i$ where $\dot{\theta}_i$, $i = 1,2,3,4.$ denotes the angular velocities for each wheel. It is assumed that wheels roll on the plane $x_u y_u$ without longitudinal and transversal slippage.
Formally, last statement in mathematical terms can be written as follows

$$A(q)\,\dot{q} = 0 \qquad (1)$$

where

$$A(q) = \begin{bmatrix} -\sin(\phi_0+\phi_1) & \cos(\phi_0+\phi_1) & c\cos\phi_1 + b\sin\phi_1 & 0 & 0 & 0 & 0 & 0 \\ -\sin(\phi_0+\phi_2) & \cos(\phi_0+\phi_2) & c\cos\phi_2 - b\sin\phi_2 & 0 & 0 & 0 & 0 & 0 \\ -\sin\phi_0 & \cos\phi_0 & -d & 0 & 0 & 0 & 0 & 0 \\ \cos(\phi_0+\phi_1) & \sin(\phi_0+\phi_1) & c\sin\phi_1 - b\cos\phi_1 & -r & 0 & 0 & 0 & 0 \\ \cos(\phi_0+\phi_2) & \sin(\phi_0+\phi_2) & c\sin\phi_2 + b\cos\phi_2 & 0 & -r & 0 & 0 & 0 \\ \cos\phi_0 & \sin\phi_0 & -b & 0 & 0 & -r & 0 & 0 \\ \cos\phi_0 & \sin\phi_0 & b & 0 & 0 & 0 & -r & 0 \end{bmatrix}$$

where $q = [x_0, y_0, \phi_0, \theta_1, \theta_2, \theta_3, \theta_4, \phi]^T$ is a vector of generalised co-ordinates.
Now using the fact that wheel axes must intersect at a point when the mobile robot turns. Thus we get

$$\tan\phi_1 = \frac{(c+d)\tan\phi}{(c+d) - b\tan\phi}$$

and
$$\tan\phi_2 = \frac{(c+d)\tan\phi}{(c+d) + b\tan\phi}$$

Constraint given by Eq. (1) imply that there exists matrix $s(q)$, which is full rank and consists of linearly independent vector fields which are spanned on the null space of matrix $A(q)$, namely

$$s(q) = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} & S_{51} & S_{61} & S_{71} & S_{81} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

where

$$S_{11} = (c+d)\cos\phi_0 - d\sin\phi_0$$
$$S_{21} = d\cos\phi_0 + (c+d)\cot\phi\sin\phi_0$$
$$S_{31} = \frac{-2b(c+d) + b^2\cot 2\phi - \cos ec\phi\,(b^2 + 2(c+d)^2)}{r(c+d-b\tan\phi)\sqrt{1+\dfrac{(c+d)^2}{(b-(c+d)\cot\phi)^2}}}$$
$$S_{41} = \frac{(c+d)^2\cos ec\phi\sec\phi + b(2(c+d) + b\tan\phi)}{r(c+d+b\tan\phi)\sqrt{1+\dfrac{(c+d)^2}{(b+(c+d)\cot\phi)^2}}}$$
$$S_{51} = \frac{-b + (c+d)\cot\phi}{r}$$
$$S_{61} = \frac{b + (c+d)\cot\phi}{r}$$
$$S_{71} = 1, S_{81} = 0$$

It is straight forward to verify that $A(q)s(q) = 0$. Since the constrained velocity is always in the null space of space of $A(q)$,there exists a pseudo velocity vector $v(t)$ such that

$$\dot{q} = s(q)v(t) \qquad (2)$$

Where n is the dimension of vector q and m is the total Number of holonomic and nonholonomic constraints imposed on the system.
Define

$$\dot{x}_0 = v_x\cos\phi_0 - v_y\sin\phi_0$$
$$\dot{y}_0 = v_x\sin\phi_0 + v_y\cos\phi_0$$

where v_x and v_y are the velocities of the centre of gravity of the mobile plateform along the x and y-axes respectively. Thus we get

$$\begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\phi}_0 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\phi_0 - \dfrac{d}{c+d}\sin\phi_0\tan\phi & 0 \\ \sin\phi_0 + \dfrac{d}{c+d}\cos\phi_0\tan\phi & 0 \\ \dfrac{-2b(c+d) + b^2\cot 2\phi - \cos ec\phi\,(b^2+2(c+d)^2)}{r(c+d-b\tan\phi)\sqrt{1+\dfrac{(c+d)^2}{(b-(c+d)\cot\phi)^2}}}\cdot\dfrac{\tan\phi}{c+d} & 0 \\ \dfrac{(c+d)^2\cos ec\phi\sec\phi + b(2(c+d)+b\tan\phi)}{r(c+d+b\tan\phi)\sqrt{1+\dfrac{(c+d)^2}{(b+(c+d)\cot\phi)^2}}}\cdot\dfrac{\tan\phi}{c+d} & 0 \\ \dfrac{-b\tan\phi + (c+d)}{r(c+d)} & 0 \\ \dfrac{b\tan\phi + (c+d)}{r(c+d)} & 0 \\ \dfrac{\tan\phi}{c+d} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ \dot{\phi} \end{bmatrix}$$

Since the control objective for the robot is to ensure that $q(t)$ tracks a reference position and orientation denoted by

$$q_d(t) = [x_d(t), y_d(t), \phi_{0d}(t)]$$

we consider only

$$\begin{bmatrix}\dot{x}_0\\ \dot{y}_0\\ \dot{\phi}_0\\ \dot{\phi}\end{bmatrix}=\begin{bmatrix}\cos\phi_0-\frac{d}{c+d}\sin\phi_0\tan\phi & 0\\ \sin\phi_0+\frac{d}{c+d}\cos\phi_0\tan\phi & 0\\ \frac{\tan\phi}{c+d} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} \quad (3)$$

where $v_1 = v_x$ and $v_2 = \dot{\phi}$

4 Dynamic Model

In this section we describe dynamics of the vehicle presented in Fig.(1). In general the dynamical model of a mobile robot is given by

$$\mathrm{M(q)\ddot{q} + V(q,\dot{q}) + G(q) + \tau_z = E(q)\tau - A^T(q)\lambda} \quad (4)$$

where $\mathrm{M(q)}$ is a symmetric, positive definite matrix, V is a centripetal and Coriolis matrix ,F is a friction vector, G is a gravity vector, τ_d is a vector of disturbances including unmodeled dynamics, E is an input transformation matrix, τ is a control input vector, A is a matrix associated with constraints, λ is a vector of constraint forces ,and $\dot{q}$ and $\ddot{q}$ denote velocity and acceleration vectors, respectively. Since in our case robot moves on a plane vector G=0.

It would be more suitable to express the dynamic equations of motion in terms of quasi velocities v_1 and v_2 .By differentiating equation (2) with respect to time we get

$$\ddot{q} = S(q)\dot{v} + \dot{S}(q)v \quad (5)$$

Next we substitute Eqs. (2) and (5) into equation (4) and multiply the resulting equation on the left hand side by matrix $s^T(q)$. After some not complicated calculations we get

$$\begin{aligned}&S^T MS\dot{v} + S^T(M\dot{S}v + V) = S^T E\tau\\ &i.e \quad \overline{M}\dot{v} + \overline{V} = \overline{\tau}\\ &where \quad \overline{M} = S^T MS \ , \ \overline{V} = S^T(M\dot{S}v + V)\\ &\qquad \overline{\tau} = S^T E\tau\end{aligned} \quad (6)$$

τ is a set of three moments $[\tau_r, \tau_l, \tau_s]^T$ in which two moments $[\tau_r, \tau_l]^T$ are acting at the wheels and one moment τ_s for steering . The matrix $\overline{M}(q)$ is which appears in equation (6) is positive definite. The matrix $\overline{M} - 2\overline{V}$ is a skew symmetric matrix.

The Lagrange formulation is used to derive the dynamic equation of motion of the mobile robot.In this case there is no gravity term in dynamic equation because the trajectory of the mobile base is constrained to the horizontal plane, i.e. Since the system can not change its vertical position , its potential energy U remains constant.

The kinetic energy of the main vehicle body i.e. the mobile plateform is

$$K_{plateform} = \frac{1}{2}m_c(\dot{x}_0^2 + \dot{y}_0^2) + \frac{1}{2}I_c\dot{\phi}_0^2$$

and the kinetic energy of the four wheels are

$$K_{wheel1} = \frac{1}{2}m_w[(\dot{x}_0^2 + \dot{y}_0^2 - 2b\dot{\phi}_0(\dot{x}_0\cos\phi_0 + \dot{y}_0\sin\phi_0) + 2c\dot{\phi}_0(\dot{y}_0\cos\phi_0 - \dot{x}_0\sin\phi_0) + b^2\dot{\phi}_0^2 + c^2\dot{\phi}_0^2] + \frac{1}{2}I_w\dot{\theta}_1^2 + \frac{1}{2}I_m\dot{\phi}_1^2$$

$$K_{wheel2} = \frac{1}{2}m_w[(\dot{x}_0^2 + \dot{y}_0^2 + 2b\dot{\phi}_0(\dot{x}_0\cos\phi_0 + \dot{y}_0\sin\phi_0) + 2c\dot{\phi}_0(\dot{y}_0\cos\phi_0 - \dot{x}_0\sin\phi_0) + b^2\dot{\phi}_0^2 + c^2\dot{\phi}_0^2] + \frac{1}{2}I_w\dot{\theta}_2^2 + \frac{1}{2}I_m\dot{\phi}_2^2$$

$$K_{wheel3} = \frac{1}{2}m_w[(\dot{x}_0^2 + \dot{y}_0^2 - 2b\dot{\phi}_0(\dot{x}_0\cos\phi_0 + \dot{y}_0\sin\phi_0) - 2d\dot{\phi}_0(\dot{y}_0\cos\phi_0 - \dot{x}_0\sin\phi_0) + b^2\dot{\phi}_0^2 + d^2\dot{\phi}_0^2] + \frac{1}{2}I_w\dot{\theta}_3^2 + \frac{1}{2}I_m\dot{\phi}_0^2$$

$$K_{wheel4} = \frac{1}{2}m_w[(\dot{x}_0^2 + \dot{y}_0^2 + 2b\dot{\phi}_0(\dot{x}_0\cos\phi_0 + \dot{y}_0\sin\phi_0) - 2d\dot{\phi}_0(\dot{y}_0\cos\phi_0 - \dot{x}_0\sin\phi_0) + b^2\dot{\phi}_0^2 + d^2\dot{\phi}_0^2] + \frac{1}{2}I_w\dot{\theta}_4^2 + \frac{1}{2}I_m\dot{\phi}_0^2$$

The total kinetic energy of the four wheeled mobile robot is given by

$$K = \frac{1}{2}m_c(\dot{x}_0^2 + \dot{y}_0^2) + \frac{1}{2}I_c\dot{\phi}_0^2 + m_w[2\{\dot{x}_0^2 + \dot{y}_0^2 + (c-d)\dot{\phi}_0(\dot{y}_0\cos\phi_0 - \dot{x}_0\sin\phi_0)\} + (2b^2 + c^2 + d^2)\dot{\phi}_0^2] + \frac{1}{2}I_w(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 + \dot{\theta}_4^2) + \frac{1}{2}I_m(\dot{\phi}_1^2 + \dot{\phi}_2^2) + I_m\dot{\phi}_0^2$$

Since there is no potential energy involved in this case, the Lagrangian of the system is given by

L = Kplatform + Kwheel 1+ Kwheel 2+ Kwheel 3 +Kwheel 4=K

The equation of motion is obtained by applying the Euler Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau - A^T(q)\lambda$$

The dynamic equation of the four wheeled mobile robot with the lagrangian multipliers

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7$$

are given by

$$\begin{aligned}&(m_c + 4m_w)\ddot{x}_0 - 2m_w[(c-d)\ddot{\phi}_0\sin\phi_0 + \dot{\phi}_0^2\cos\phi_0]\\ &= \lambda_1\sin(\phi_0+\phi_1) + \lambda_2\sin(\phi_0+\phi_2) + \lambda_3\sin\phi_0 -\\ &\lambda_4\cos(\phi_0+\phi_1) - \lambda_5\cos(\phi_0+\phi_2) - (\lambda_6+\lambda_7)\cos(\phi_0).\end{aligned} \quad (7)$$

$$\begin{aligned}&(m_c + 4m_w)\ddot{y}_0 + 2m_w[(c-d)\ddot{\phi}_0\cos\phi_0 - \dot{\phi}_0^2\sin\phi_0]\\ &= -\lambda_1\cos(\phi_0+\phi_1) - \lambda_2\cos(\phi_0+\phi_2) - \lambda_3\cos\phi_0 -\\ &\lambda_4\cos(\phi_0+\phi_1) - \lambda_5\sin(\phi_0+\phi_2) - (\lambda_6+\lambda_7)\sin(\phi_0).\end{aligned} \quad (8)$$

$$\begin{aligned}&I_c\ddot{\phi}_0 + m_w[2(c-d)(\ddot{y}_0\cos\phi_0 - \ddot{x}_0\sin\phi_0) + (2b^2 +\\ &c^2 + d^2)\ddot{\phi}_0] + 2I_m\ddot{\phi}_0 = -\lambda_1(c\cos\phi_1 + b\sin\phi_1) -\\ &\lambda_2(c\cos\phi_2 - b\sin\phi_2) + d\lambda_3 - (c\sin\phi_1 - b\cos\phi_1)\lambda_4\\ &-(c\sin\phi_2 + b\cos\phi_2)\lambda_5 + b(\lambda_6 - \lambda_7)\end{aligned} \quad (9)$$

$$I_w\ddot{\theta}_1 = r\lambda_4 \ldots\ldots\ldots\ldots\ldots \quad (10)$$

$$I_w\ddot{\theta}_2 = r\lambda_5 \ldots\ldots\ldots\ldots\ldots \quad (11)$$

$$I_w\ddot{\theta}_3 = \tau_3 + r\lambda_6 \ldots\ldots\ldots\ldots \quad (12)$$

$$I_w\ddot{\theta}_4 = \tau_4 + r\lambda_7 \ldots\ldots\ldots\ldots\ldots \quad (13)$$

$$f(\phi)\ddot{\phi} + \frac{1}{2}\dot{f}(\phi)\dot{\phi}^2 = \tau_s \quad (14)$$

where

$$f(\phi)=\frac{(c+d)^4 Sec^6\phi(b^2+2(c+d)^2-b^2Cos2\phi-2b(c+d)Sin2\phi)}{[(c+d)^2-2b(c+d)\tan\phi+(b^2+2(c+d)^2)\tan^2\phi]^3}$$

$$\dot f(\phi)=\frac{(c+d)^4 Sec^6\phi(-2b^2Sin2\phi+4b(c+d)Cos2\phi)}{[(c+d)^2-2b(c+d)\tan\phi+(b^2+2(c+d)^2)\tan^2\phi]^3}$$

The above dynamic equation of motion for the four wheel nonholonomic mobile robot can be written in the form

$$M(q)\ddot q + C(q,\dot q) = E(q)\tau - A^T(q)\lambda$$

where

$$M(q)=\begin{bmatrix} m_c+4m_w & 0 & -2m_w(c-d)\sin\phi_0 & 0 & 0 & 0 & 0 & 0\\ 0 & m_c+4m_w & 2m_w(c-d)\cos\phi_0 & 0 & 0 & 0 & 0 & 0\\ -2m_w(c-d)\sin\phi_0 & 2m_w(c-d)\cos\phi_0 & I_c+m_w(2b^2+c^2+d^2)+2I_m & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & I_w & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I_w & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & I_w & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & I_w & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & f(\phi) \end{bmatrix}$$

$$V(q,\dot q)=\begin{bmatrix} -2m_w\dot\phi_0^2\cos\phi_0\\ -2m_w\dot\phi_0^2\sin\phi_0\\ 0\\ 0\\ 0\\ 0\\ 0\\ \frac{1}{2}\dot f(\phi)\dot\phi^2 \end{bmatrix}, E(q)=\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \tau=\begin{bmatrix}\tau_3\\ \tau_4\\ \tau_s\end{bmatrix}, \lambda=\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\\ \lambda_5\\ \lambda_6\\ \lambda_7\end{bmatrix}$$

and the constraint matrix is given by (1).

5 Chained form of kinematic model

Using the following change of co-ordinates

$$\begin{aligned} x_1 &= x_0 - d\cos\phi_0\\ x_2 &= \frac{\tan\phi}{(c+d)\cos^3\phi_0}\\ x_3 &= \tan\phi_0\\ x_4 &= y_0 - d\sin\phi_0 \end{aligned}$$

the kinematic model (3) can be transformed in the following chained form

$$\begin{aligned} \dot x_1 &= u_1\\ \dot x_2 &= u_2\\ \dot x_3 &= x_2 u_1\\ \dot x_4 &= x_3 u_1 \end{aligned} \tag{15}$$

with two input transformations

$$v_1=\frac{u_1}{\cos\phi_0}$$

$$v_2=\frac{-3\sin^2\phi\sin\phi_0}{(c+d)\cos^2\phi_0}u_1+(c+d)\cos^3\phi_0\cos^2\phi\ u_2$$

above is the two input four state chained form where $x=(x_1,x_2,x_3,x_4)$ is the state and u_1 and u_2 are the two control inputs.

5.1 Reference trajectory generation

Assuming that the reference state trajectory and reference input trajectory for the four wheel mobile robot be $q_d(t)=(x_{d0}(t), y_{d0}(t), \phi_{0d}(t), \phi_d(t))$, $V_d(t)=(V_{d1}(t), V_{d2}(t))$

The desired trajectory is feasible only when it satisfies the nonholonomic constraints on the system.
Given any boundary conditions

$$x(t_0)=[x_1(0), x_2(0), x_3(0), x_4(0)]$$

and

$$x(t_d)=[x_{d1}, x_{d2}, x_{d3}, x_{d4}]^T \quad \text{for some } t_0 > t_d .$$

There exists inputs v_1 and v_2 that gives a feasible trajectory of functional form

$$x_4 = F(x_1)$$

Satisfying the following boundary conditions

$$x_4(0)=F[x_1(0)], x_{4d}=F[x_1(d)]$$

$$x_3(0)=\frac{dF(x_1(0))}{dx_1(0)}, x_3(d)=\frac{dF(x_1(d))}{dx_1(d)}$$

$$x_2(0)=\frac{d^2F(x_1(0))}{dx_1^2(0)}, x_2(d)=\frac{d^2F(x_1(d))}{dx_1^2(d)}$$

Since there are six boundary condition .So minimum order of polynomial type feasible trajectory is five. So we assume that

$$\begin{aligned} x_4 &= F(x_1)\\ &= a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 + a_4x_1^4 + a_5x_1^5 \end{aligned}$$

6 Design of tracking controller

Denote the tracking error as $x_e = x - x_d$ .The error differential equation are

$$\begin{aligned} \dot x_{e1} &= u_1 - u_{d1}\\ \dot x_{e2} &= u_2 - u_{d2}\\ \dot x_{e3} &= x_{e2}u_{d1} + x_2(u_1-u_{d1})\\ \dot x_{e4} &= x_{e3}u_{d1} + x_3(u_1-u_{d1}) \end{aligned} \tag{16}$$

The goal is to find a time-varying controller,

$$u=\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\tilde u(x_e, u_{1d}, u_{2d})$$

such that the tracking error x_e converges to zero asymptotically, i.e. $\lim_{t\to\infty}\|x-x_d\|=0$ under appropriate conditions on the reference control functions u_{1d} and

u_{2d} and initial tracking errors $x_e(0)$, with a good choice of λ.

We first introduce a change of coordinates and rearrange system (16) into a triangular –like form so that the integrator backstepping can be applied.

Denote $\tilde{x}_d = (x_{2d}, x_{3d})$ and let $\eta_1(.;\tilde{x}_d): R^4 \to R^4$ be the mapping defined by

$$\zeta_i = x_{e(4-i+1)} - (x_{e(n-i)} + x_{d(n-i)})x_{e1}, \quad 1 \le i \le 2$$
$$\zeta_3 = x_{2e}$$
$$\zeta_4 = x_{1e}$$

In the new coordinates $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ system (16) is transformed into

$$\begin{aligned}\dot{\zeta}_1 &= u_{d1}\zeta_2 - x_2(u_1 - u_{d1})\zeta_4\\ \dot{\zeta}_2 &= u_{d1}\zeta_3 - u_2\zeta_4\\ \dot{\zeta}_3 &= u_2 - u_{d2}\\ \dot{\zeta}_4 &= u_1 - u_{d1}\end{aligned} \tag{17}$$

Consider the ζ_1 – subsystem of (17)

$$\dot{\zeta}_1 = u_{d1}\zeta_2 - x_2(u_1 - u_{d1})\zeta_4 \tag{18}$$

We consider the variable ζ_2 as virtual control input and the variable u_{d1} and ζ_4 as time varying functions.

Denote $\overline{\zeta_1} = \zeta_1$. Differentiating the function $V_1 = \frac{1}{2}\overline{\zeta_1}^2$ along the solution of (17) yields

$$\dot{V}_1 = u_{d1}\overline{\zeta_1}\zeta_2 - x_2\overline{\zeta_1}(u_1 - u_{d1})\zeta_4 \tag{19}$$

Observe that $\alpha_1(\zeta_1) = 0$ is a stabilizing function for system (7) whenever $\zeta_4 = 0$.

Define $\overline{\zeta_2} = \zeta_2 - \alpha_1(\zeta_1)$

Differentiating the function

$$V_2 = V_1 + \frac{1}{2}\overline{\zeta_2}^2 = \frac{1}{2}\overline{\zeta_1}^2 + \frac{1}{2}\overline{\zeta_2}^2$$

along the solution of (6) yields

$$\dot{V}_2 = u_{d1}\overline{\zeta_2}\,\overline{\zeta_3} - x_2\overline{\zeta_1}(u_1 - u_{d1})\zeta_4 - u_2\overline{\zeta_2}\zeta_4$$

As

$$\overline{\zeta_2} = \zeta_2 - \alpha_1(\zeta_1)$$
$$\dot{\overline{\zeta}}_2 = \dot{\zeta}_2 - \frac{\partial\alpha_1}{\partial\zeta_1}\dot{\zeta}_1$$
$$\begin{aligned}&= \dot{\zeta}_2 = u_{d1}\zeta_3 - u_2\zeta_4\\ &= u_{d1}\zeta_3 - u_2\zeta_4 + u_{d1}\zeta_1 - u_{d1}\zeta_1\\ &= u_{d1}(\zeta_3 + \zeta_1) - u_2\zeta_4 - u_{d1}\zeta_1\\ &= u_{d1}(\zeta_3 - \alpha_2) - u_2\zeta_4 - u_{d1}\zeta_1\end{aligned}$$

where

$$\alpha_2(\zeta_1, \zeta_2) = -\zeta_1$$

Again define

$$\overline{\zeta_3} = \zeta_3 - \alpha_2(\zeta_1, \zeta_2)$$
$$\dot{\overline{\zeta}}_3 = \dot{\zeta}_3 - \left(\frac{\partial\alpha_2}{\partial\zeta_1}\dot{\zeta}_1 + \frac{\partial\alpha_2}{\partial\zeta_2}\dot{\zeta}_2\right)$$
$$\begin{aligned}&= \dot{\zeta}_3 - (-\dot{\zeta}_1) = \dot{\zeta}_3 + \dot{\zeta}_1\\ &= u_2 - u_{d2} + u_{d1}\zeta_2 - x_2(u_1 - u_{d1})\zeta_4\end{aligned}$$

Consider the positive definite and proper function

$$V_3 = V_2 + \frac{1}{2}\overline{\zeta_3}^2 = \frac{1}{2}\overline{\zeta_1}^2 + \frac{1}{2}\overline{\zeta_2}^2 + \frac{1}{2}\overline{\zeta_3}^2$$

Differentiating the function V_3 along the solution of (17) yields

$$\begin{aligned}\dot{V}_3 = {}&\overline{\zeta_3}(u_{d1}\overline{\zeta_2} + u_2 - u_{d2} + u_{d1}\zeta_2)\\ &- (x_2\overline{\zeta_1} + x_2\overline{\zeta_3})(u_1 - u_{d1})\zeta_4 - u_2\overline{\zeta_2}\zeta_4\end{aligned}$$

In order to make V_3 negative definite we choose the following control input

$$u_{d1}\overline{\zeta_2} + u_2 - u_{d2} + u_{d1}\zeta_2 = -c_3\overline{\zeta_3} \tag{20}$$

where $c_3 > 0$. Thus we have

$$\dot{V}_3 = -c_3\overline{\zeta_3}^2 - (x_2\overline{\zeta_1} + x_2\overline{\zeta_3})(u_1 - u_{d1})\zeta_4 - u_2\overline{\zeta_2}\zeta_4$$

Finally consider the positive definite and proper function which serves as a candidate Lyapunov function for the whole system (17)

$$V_4 = V_3 + \frac{\lambda}{2}\zeta_4^2 = \frac{1}{2}\overline{\zeta_1}^2 + \frac{1}{2}\overline{\zeta_2}^2 + \frac{1}{2}\overline{\zeta_3}^2 + \frac{\lambda}{2}\zeta_4^2$$

where $\lambda > 0$ is a design parameter.

Differentiating the function V_4 along the solution of (17) yields

$$\begin{aligned}\dot{V}_4 = {}&-c_3\overline{\zeta_3}^2 - (x_2\overline{\zeta_1} + x_2\overline{\zeta_3})(u_1 - u_{d1})\zeta_4 - u_2\overline{\zeta_2}\zeta_4 +\\ &\lambda\zeta_4(u_1 - u_{d1})\end{aligned}$$
$$\dot{V}_4 = -c_3\overline{\zeta_3}^2 - [\{\lambda - (x_2\overline{\zeta_1} + x_2\overline{\zeta_3})\}(u_1 - u_{d1}) - u_2\overline{\zeta_2}]\zeta_4$$

In order to make V_4 negative definite we choose the following control input

$$\{\lambda - (x_2\overline{\zeta_1} + x_2\overline{\zeta_3})\}(u_1 - u_{d1}) - u_2\overline{\zeta_2} = c_4\zeta_4 \tag{21}$$

From (9) and (10) we get the following control law

$$u_1 = u_{d1} + \frac{-c_4\zeta_4 + u_2\zeta_2}{\lambda - (2x_2\zeta_1 + x_2\zeta_3)}$$
$$u_2 = u_{d2} - c_3(\zeta_3 + \zeta_1) - 2u_{d1}\zeta_2$$

7 Dynamic based controller

Since S is a full rank matrix formed by a set of smooth and linearly independent vector field .we have from equation (6) , the reduced mass matrix $\overline{M}$ is always symmetric and positive definite.

Thus we have

$$\overline{\tau} = \overline{M}\dot{v} + \overline{V}$$

Now define the backstepping error e as

$$e = v - v_d$$

Selecting the following a Lyapunov candidate function V_1 for the mobile plateform

$$V_1 = \frac{1}{2} e^T \overline{M} e$$

The time derivative of V_1 along the system trajectory is

$$\dot{V}_1 = \frac{1}{2} e^T \dot{\overline{M}} e + e^T \overline{M} \dot{e}$$

Since $\quad x^T(\dot{\overline{M}} - 2C)x = 0 \;\forall \quad x \neq 0.$

$$\dot{V}_1 = \frac{1}{2} e^T (\dot{\overline{M}} - 2\overline{V})e + e^T \overline{V} e + e^T \overline{M} \dot{e}$$
$$= e^T \overline{V} e + e^T \overline{M} \dot{e}$$
$$\overline{\tau} = \overline{M}(\dot{e} + \dot{v}_d) + \overline{V}$$

Hence

$$\dot{V}_1 = e^T(\overline{V} e + \overline{\tau} - \overline{V} - \overline{M}\dot{v}_d)$$
$$= e^T(\overline{V} v - \overline{V} \dot{v}_d + \overline{\tau} - \overline{V} - \overline{M}\dot{v}_d)$$

Choosing the control law as

$$\overline{\tau} = \overline{M} u + \phi$$

Where $\quad \phi = -\overline{V} v + \overline{V} v_d + \overline{V},\; u = \dot{v}_d - ke$

Then $\quad \dot{V}_1 = -k\, e^T \overline{M} e$

Thus the derivative of V_1 is negative definite. Hence the system is asymptotically stable.

8 Simulation results

To examine the effectiveness of the proposed trajectory tracking control methodology, the simulation for a four wheeled mobile robot were performed in MATLAB and MATHEMATICA. The system parameters of the four wheel mobile robot were selected as

c=1.3m, d=1.4 m, 2b=1.5m, Mc=1000kg ,
v=10m/s,Ic=795kg m^2, Mw=20kg , r=0.2m ,
Im=0.20 kg m^2, Iw=0.40 kg m^2.

We consider the following reference output trajectory $x_d(t) = t$, $y_d(t) = \sin t$ with the conditions

$$x_0(0) = 0,\; y_0(0) = 0,\; \phi_0(0) = \frac{\pi}{4}, \phi(0) = 0, x_d(40) = 17, y_d(40) = 10,\; \phi_{0d}(40) = -\frac{\pi}{4},\quad \phi(40) = 0.$$

Fig.9 demonstrates the evolution of the norm of the tracking error $x_e(t)$ based on the following choice of design parameters and initial condition:

$$\lambda = 3, c_3 = c_4 = 5,\; x_e(0) = (2, 0.5, 0.5, 1.5)$$

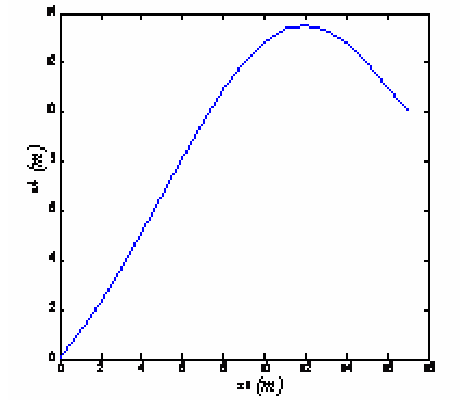

Figure 3: Feasible trajectory

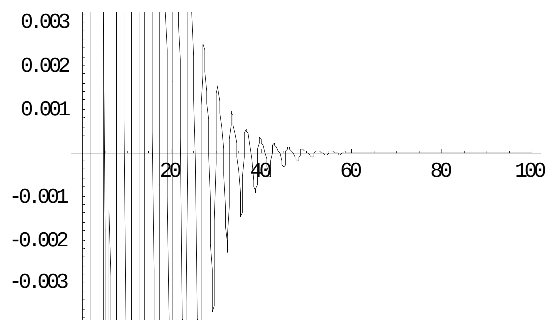

Figure 4: Tracking error in $x_1(t)$

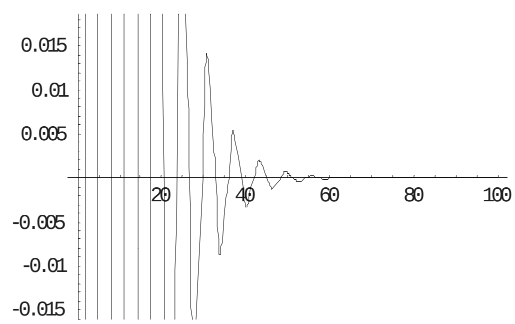

Figure 5: Tracking error in $x_2(t)$

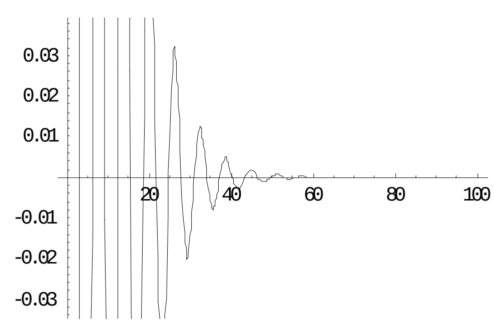

Figure 6: Tracking error in $x_3(t)$

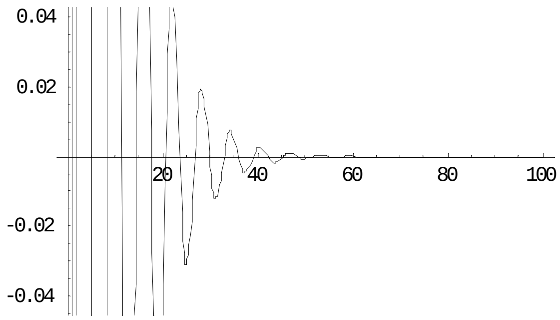

Figure 7: Tracking error in $x_4(t)$

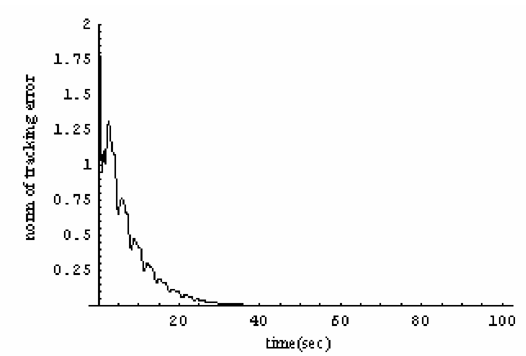

Figure 8: norm of tracking error $x_e(t)$

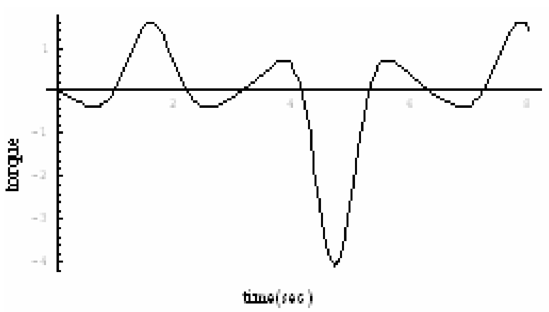

Figure 9: Driving wheel torque τ_s

5 CONCLUSION

In this paper the nonholonomic constraints and the kinematics model of the four wheel (front steering and rear driving) mobile robot under pure rolling and no side slipping condition is derived .Using the change of coordinates the system is transformed into chained form and then a backstepping based tracking controller is derived. Simulation results are presented with two examples to illustrate the approach.

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